Journal of Geometry and Physics 8 (1992) 211–219
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

The linear system for self-dual gauge fields in a spacetime of signature 0

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Received 9 May 1991

The overdetermined linear system for the self-dual Yang–Mills (SDYM) equations is examined in a flat four-dimensional space whose metric has signature 0. There are three different domains for the system, and correspondingly three (essentially) different solutions to the linear system for a given gauge field. If the gauge potential is real analytic, two of the solutions patch together to give a holomorphic function in an annular region of projective twistor space. Conversely, an *arbitrary* holomorphic $GL(n, \mathbb{C})$-valued function in such a domain can be uniquely factored (on the real lines) to give a solution to SDYM with gauge group $U(n)$. The set of all real analytic $u(n)$-valued gauge fields can thus be parametrized by the points of a certain double coset space.

Keywords: *Self-dual gauge field*
1991 MSC: 81T13

As Richard Ward showed a number of years ago [1], the study of real analytic self-dual gauge fields in four dimensions is equivalent to the study of certain holomorphic vector bundles on domains in $P_3(\mathbb{C})$. Analyticity is actually a consequence of self-duality in the positive-definite case [2], in the sense that one can always construct a gauge in which the connection is real analytic. For the other signatures, non-analytic solutions are generic, so Ward's construction is not always possible. In this paper, we examine some properties of the self-dual Yang–Mills (SDYM) fields in the flat "spacetime" M with metric $\mathrm{diag}(1, 1, -1, -1)$. The main interest in these fields is that, in the presence of certain symmetries, they yield solutions to a number of important integrable systems. In particular, solutions to the KdV and non-linear Schrödinger equations [3], the sine-Gordon equation, certain harmonic maps and non-linear σ-models are all obtained by reduction [4] of the self-duality equations in M. We shall argue below that the full SDYM equations themselves possess an inherently rich mathematical structure.

Recall [1] that the self-duality equations are the integrability conditions for the existence of solutions to the overdetermined linear system $G^{-1}D_AG=\Gamma_A$, where the operators $D_A(=\pi^{A'}\nabla_{AA'})$ contain an auxiliary spectral parameter taking values on the Riemann sphere (see sections 2, 3). Due to the local isomorphism $SO(2,2)\approx SU(1,1)\times SU(1,1)$, the group $SU(1,1)$ plays a fundamental role in the case of zero signature. Here the Riemann sphere decomposes into the disjoint union $D_0+S^1+D_\infty$, of orbits of this group, where D_0 and D_∞ are the interior and exterior of the unit circle S^1. This has the consequence that there are three disjoint domains, $M\times D_0$, $M\times S^1$, and $M\times D_\infty$, for the linear system mentioned above, rather than the more usual (single) bundle of projective spinors $M\times S^2$.

More precisely, the linear system on $M\times S^1$ (section 2) is a natural generalization of the extended linear system for harmonic maps [5,6]. It is solved in a straightforward way, by constructing parallel propagators in each of the real anti-self-dual two-planes in M, thus giving a totally real version of Ward's original construction.

The bundles $M\times D_{0,\infty}$ can be identified with non-holomorphic fibrations of open subsets $P_{0,\infty}$ of $P_3(\mathbb{C})$ by unit disks (section 1). The construction of solutions to the corresponding linear system [by pulling back the complex structure of $P_3(\mathbb{C})$] follows that of Atiyah et al. [2] (section 3).

In the real analytic case, of course, one obtains much stronger results (section 4). Here the domains of the two constructions overlap, and the two different solutions to the linear system determine a non-singular holomorphic matrix function in an "annular" region of P_0.

Conversely, specializing to the gauge group $U(n)$, we find that an *arbitrary* $GL(n,\mathbb{C})$-valued holomorphic function defined in such a domain of P_0 factors uniquely in each unit disk to give a real analytic solution to SDYM. The usual Riemann–Hilbert problem does not appear here. In fact, none of the solutions to the linear system discussed in this paper is obtained directly by splitting the transition functions for a holomorphic bundle. But there is a different splitting which allows us to recover the gauge field in essentially the same manner.

The group G of all such matrix-valued holomorphic functions is then mapped (by the factorization) onto the set of solutions to the linear system and hence onto the set of solutions to the SDYM equations. More precisely, the extended solutions of section 2 are parametrized by the points of a certain homogeneous space G/G_+, and the self-dual gauge fields themselves by the points of a double coset space $G_{\mathbb{R}}\backslash G/G_+$.

1. The Penrose correspondence for M

Let M denote $\mathbb{R}^4$ with the indefinite metric $\mathrm{d}s^2+\mathrm{d}t^2-\mathrm{d}u^2-\mathrm{d}v^2$. In the complex coordinates $y=s+\mathrm{i}t$, $z=u+\mathrm{i}v$, this becomes $\mathrm{d}y\,\mathrm{d}\bar{y}-\mathrm{d}z\,\mathrm{d}\bar{z}$, and a basis for the anti-self-dual (ASD) two-forms is given by $\{\mathrm{d}y\wedge\mathrm{d}z, \mathrm{d}\bar{y}\wedge\mathrm{d}\bar{z}, \mathrm{d}y\wedge\mathrm{d}\bar{z}-\mathrm{d}z\wedge\mathrm{d}\bar{z}\}$.

The SDYM equations are $F_{yz}=F_{\bar{y}\bar{z}}=0$, $F_{y\bar{y}}-F_{z\bar{z}}=0$.

For each triple (a, b, ϑ) in $\mathrm{R}^2\times\mathrm{S}^1$ the points $x=(s, t, u, v)\in M$ satisfying

$$s=a-u\cos\vartheta-v\sin\vartheta\,,\qquad t=b-u\sin\vartheta+v\cos\vartheta\,,$$

form a real ASD two-plane. In terms of the complex coordinates, the equation is $y=y_0-\lambda\bar{z}$, where $\lambda=\mathrm{e}^{\mathrm{i}\vartheta}$ lies on the unit circle. Taking differentials, $\mathrm{d}y+\lambda\,\mathrm{d}\bar{z}=0=\mathrm{d}\bar{y}+\lambda^{-1}\mathrm{d}z$, where the two-form $(1/2\mathrm{i})(\mathrm{d}y+\lambda\,\mathrm{d}\bar{z})\wedge(\mathrm{d}\bar{y}+\lambda^{-1}\mathrm{d}z)$ is ASD, real, and vanishes on the two-plane. Alternatively, the real and imaginary parts of $\mathrm{d}y+\lambda\,\mathrm{d}\bar{z}=0$ determine the ASD two-plane.

In this signature, the Penrose correspondence [7] associates to each point $x=(y, \bar{y}, z, \bar{z})\in M$ the projective line

$$L_x=\{(v, w, \lambda): v=y+\lambda\bar{z},\ w=z+\lambda\bar{y},\ \lambda\in\mathrm{P}_1(\mathbb{C})\} \tag{1}$$

in $\mathrm{P}_3(\mathbb{C})$. The expression given above is for the affine chart $(Z_\alpha)=(v, w, 1, \lambda)$. Recall that $L_x\cap L_{x'}\neq\emptyset$ exactly when x and x' are null separated, and notice that for $x, x'\in M$, the point of intersection, if it exists, necessarily occurs for some $\lambda=\lambda'$ lying on the unit circle $|\lambda|=1$.

Thus the conformal structure of M is completely determined by the *real* ($\approx\mathrm{S}^1$) projective lines $\{L_x|_{|\lambda|=1}, x\in M\}$. Moreover, given x and a fixed λ on the unit circle, the line $L_{x+\mathrm{d}x}$ intersects L_x at the point labeled by λ if and only if $\mathrm{d}y+\lambda\,\mathrm{d}\bar{z}=0$. Now the set pN of points (v, w, λ) with $|\lambda|=1$ that actually lie on lines corresponding to the points of M is three (real) dimensional, since $v=y+\lambda\bar{z}$, $w=z+\lambda\bar{y}\Rightarrow w=\lambda\bar{v}$. Comparing this with the above gives a 1–1 correspondence between pN and the set of (finite) ASD two-planes in M. In fact, it is not difficult to check that pN is just $\mathrm{P}_3(\mathbb{R})\setminus\mathrm{P}_1(\mathbb{R})$, where $\mathrm{P}_3(\mathbb{R})$ is the totally real twistor space associated to M, and the deleted $\mathrm{P}_1(\mathbb{R})$ is the equator of the line corresponding to the vertex of the null cone at infinity. See, for example, the appendix in Penrose and Rindler [7].

We regard the correspondence as a map p from the trivial bundle $M\times\mathrm{S}^2$ into $\mathrm{P}_3(\mathbb{C})\setminus\{Z_2=Z_3=0\}$:

$$p:(y, \bar{y}, z, \bar{z}, \lambda)\rightarrow(y+\lambda\bar{z}, z+\lambda\bar{y}, 1, \lambda)\,. \tag{2}$$

The image of $M\times\mathrm{S}^1$ is pN of course, and we denote the images of $M\times D_0$, $M\times D_\infty$ by P_0, P_∞, respectively. The map p is a real analytic bijection of $M\times\mathrm{S}^2\setminus M\times\mathrm{S}^1$ onto $P_0\cup P_\infty$; its inverse is given by

$$y=\frac{v-\lambda\bar{w}}{1-|\lambda|^2}\,,\qquad z=\frac{w-\lambda\bar{v}}{1-|\lambda|^2}\,,\quad \lambda=\lambda\,. \tag{3}$$

We may restrict our attention to $P_0\approx\mathbb{C}^2\times D_0=\{(v, w, \lambda):|\lambda|<1\}$. (With the obvious changes, every statement about P_0 can be converted into one about P_∞.) As we saw above, for any distinct points x, x' in M, $L_x|P_0\cap L_{x'}|P_0=\emptyset$. In fact, P_0 is fibered (non-holomorphically) over M by these unit disks, the projection

$\tau: P_0 \to M$ being given by

$$\left(y = \frac{v - \lambda\bar{w}}{1 - |\lambda|^2}, z = \frac{w - \lambda\bar{v}}{1 - |\lambda|^2}\right), \quad |\lambda| < 1 .$$

The natural projection $\pi: M \times D_0 \to M$ which sends (y, z, λ) to (y, z) is holomorphic, and $\pi = \tau p$. But neither τ nor p is holomorphic, although $\tau^{-1}(x)$ is the holomorphic "curve" $L_x \cap P_0$.

2. The linear system for $|\lambda| = 1$

In the next two sections, we consider what can be said about non-analytic solutions to the SDYM equations in M. We suppose a gauge exists in which the connection is C^∞, although this condition can evidently be relaxed. Recall that the ASD two-planes in M are given by $dy + \lambda\, d\bar{z} = 0$, where λ is some fixed point on the unit circle. If for each such λ we introduce the complex coordinates $\zeta(\lambda) = z - \lambda\bar{y}$, $\eta(\lambda) = y + \lambda\bar{z}$, the ASD two-planes are then given by the complex lines $\eta(\lambda) =$ constant as λ varies over the unit circle.

Suppose now that A is a C^∞ connection in the trivial n-plane bundle on M with self-dual curvature $F = dA + A \wedge A$. Fix a point $(x, \lambda) \in M \times S^1$. Then x lies in a unique ASD two-plane $\eta(\lambda) =$ const., and we can define $\Psi(x, \lambda) := P(\exp \int A)$, where the parallel propagator is evaluated along any curve in the two-plane going from x to the point $\zeta(\lambda) = 0$. Independence of the path of integration follows from self-duality.

The function Ψ then satisfies the equations $\Psi^{-1}[\partial/\partial\zeta(\lambda)]\Psi = A_{\zeta(\lambda)}$, $\Psi^{-1}[\partial/\partial\bar{\zeta}(\lambda)]\Psi = A_{\bar{\zeta}(\lambda)}$, corresponding to the fact that A is a pure gauge in these two-planes. Rewriting this in terms of y and z, we get a solution to the linear system

$$\Psi^{-1}(\partial/\partial z - \lambda^{-1}\partial/\partial\bar{y})\Psi = A_z - \lambda^{-1}A_{\bar{y}}, \tag{4a}$$

$$\Psi^{-1}(\partial/\partial\bar{z} - \lambda\,\partial/\partial y)\Psi = A_{\bar{z}} - \lambda A_y . \tag{4b}$$

Conversely, it is well known and easily checked that the integrability conditions for the overdetermined linear system (4) are precisely the SDYM equations. If the gauge group is $U(n)$, then A is skew-Hermitian and the second equation is minus the adjoint of the first. The system is still overdetermined, however, because of the prescribed λ-dependence of the right-hand side. We can fix gauges uniquely by setting $\Phi(x, \lambda) = \Psi(x, \lambda)\,\Psi^{-1}(x, 1)$. Then $\Phi(x, 1) \equiv I$, and the right-hand side of (4) vanishes at $\lambda = 1$. In this gauge, the equations take the simpler form

$$\Phi^{-1}(\partial/\partial z - \lambda^{-1}\partial/\partial\bar{y})\Phi = (1 - \lambda^{-1})B , \tag{5a}$$

$$\Phi^{-1}(\partial/\partial\bar{z} - \lambda\,\partial/\partial y)\Phi = (1 - \lambda)C , \tag{5b}$$

where B and C are matrices depending on x alone. If A is skew-Hermitian, then $C=-B^*$, and $\Phi(\lambda, x)$ is a smooth map from M to the based unitary loop group $\Omega\mathrm{U}(n)$, called an *extended solution* to the SDYM equations. Setting $\varphi(x) := \Phi(x, -1)$ gives a smooth map of M into the group $\mathrm{U}(n)$ called a generalized harmonic map [6,8]. In particular, if φ is independent of y and $\bar{y}$, then it is harmonic in the usual sense [5], but we shall not pursue this any further here.

Remark. Although the gauge of $F=\mathrm{d}A+A\wedge A$ is fixed by the requirement $\Phi(x, 1)\equiv I$, the function $\Phi(x, \lambda)$ is certainly not unique. For we can replace it by $\gamma\Phi$, where γ is any smooth $\mathrm{GL}(n, \mathbb{C})$-valued function which is annihilated by the two differential operators in (3). That is, $\gamma(x, \lambda)=g(p(x, \lambda))$, where g is smooth on pN and $g|_{\lambda=1}\equiv I$. The freedom here corresponds to smoothly varying the choice of origin $\zeta(\lambda)=0$ in each ASD two-plane. We shall return to this point in section 5.

3. The linear system for $|\lambda|<1$

The first observation here is (cf. Atiyah et al. [2])

Proposition 1. *If F is self-dual on M, then τ^*F is of type (1,1) in the complex structure of P_0.*

Proof. A routine computation gives

$$\tau^*(\mathrm{d}y\wedge \mathrm{d}\bar{y}+\mathrm{d}z\wedge \mathrm{d}\bar{z}) = (1-|\lambda|^2)[\mathrm{d}v\wedge \mathrm{d}\bar{v}+\mathrm{d}w\wedge \mathrm{d}\bar{w} \\ -z\,\mathrm{d}v\wedge \mathrm{d}\bar{\lambda}-\bar{y}\,\mathrm{d}\lambda\wedge \mathrm{d}\bar{w}-(y\bar{y}-z\bar{z})\mathrm{d}\bar{\lambda}\wedge \mathrm{d}\lambda]\,.$$

Similarly $\tau^*(\mathrm{d}y\wedge \mathrm{d}\bar{z})$ and $\tau^*(\mathrm{d}\bar{y}\wedge \mathrm{d}z)$ are of type (1, 1). Since these span the self-dual two-forms in M, the result follows. □

Suppose now that A is a self-dual potential, and write $\mathscr{A}:=\tau^*A$ as the sum $\mathscr{A}'+\mathscr{A}''$ of forms of type (1, 0) and (0, 1), respectively. The proposition implies that $\bar{\partial}\mathscr{A}''+\mathscr{A}''\wedge\mathscr{A}''=0$. By a theorem of Malgrange [9] we can find an open cover $\{U_a\}$ of P_0 and smooth functions $h_a: U_a\to\mathrm{GL}(n, \mathbb{C})$ such that $h_a^{-1}\bar{\partial}h_a= \mathscr{A}''$ in U_a. The functions $g_{ab}:=h_a h_b^{-1}$, defined whenever $U_a\cap U_b\neq\emptyset$, are then holomorphic ($\bar{\partial}g_{ab}=0$). Since P_0 is Stein and topologically trivial, a theorem of Grauert [10] guarantees the existence of holomorphic functions $c_a: U_a\to \mathrm{GL}(n, \mathbb{C})$ satisfying $g_{ab}=c_a^{-1}c_b$. Then $h:=c_a h_a=c_b h_b$ is a *global* smooth function from P_0 to the general linear group such that $h^{-1}\bar{\partial}h=\mathscr{A}''$.

Theorem 2. *For A and h as above, let $H(x, \lambda)=p^*h$. Then H is holomorphic in λ for $|\lambda|<1$ and solves the linear system*

$$H^{-1}(\partial/\partial\bar{y}-\lambda\,\partial/\partial z)H=A_{\bar{y}}-\lambda A_z\,, \tag{6a}$$

$$H^{-1}(\partial/\partial\bar{z}-\lambda\,\partial/\partial y)H=A_{\bar{z}}-\lambda A_y\,. \tag{6b}$$

Proof.

$$\mathscr{A}''=A_{\bar{v}}\,\mathrm{d}\bar{v}+A_{\bar{w}}\,\mathrm{d}\bar{w}+A_{\bar{\lambda}}\,\mathrm{d}\bar{\lambda}\,,$$

where

$$h^{-1}\partial_{\bar{v}}h=A_{\bar{v}}=(1-|\lambda|^2)^{-1}(A_{\bar{y}}-\lambda A_z)\,,$$

$$h^{-1}\partial_{\bar{w}}h=A_{\bar{w}}=(1-|\lambda|^2)^{-1}(A_{\bar{z}}-\lambda A_y)\,,$$

$$h^{-1}\partial_{\bar{\lambda}}h=A_{\bar{\lambda}}=(1-|\lambda|^2)^{-1}\{-z(A_{\bar{y}}-\lambda A_z)-y(A_{\bar{z}}-\lambda A_y)\}\,.$$

Now

$$\partial_{\bar{v}}=(1-|\lambda|^2)^{-1}(\partial_{\bar{y}}-\lambda\,\partial_z)\,,$$

$$\partial_{\bar{w}}=(1-|\lambda|^2)^{-1}(\partial_{\bar{z}}-\lambda\,\partial_y)\,,$$

so the first two of these are equivalent to (6) above. On the other hand, given the form of $A_{\bar{v}}$ and $A_{\bar{w}}$, we may rewrite the third equation as

$$h^{-1}(\partial_{\bar{\lambda}}h+z\,\partial_{\bar{v}}h+y\,\partial_{\bar{w}}h)=0\,.$$

Since $\det(h)\neq 0$, this gives $(\partial_{\bar{\lambda}}+z\,\partial_{\bar{v}}+y\,\partial_{\bar{w}})h=0$. But this is exactly the equation $\partial H/\partial\bar{\lambda}=0$, so H is holomorphic in λ. □

Remarks.

(1) Away from $\lambda=0$, we can multiply (6a) by $-1/\lambda$ to get a system which is formally identical to (4). The λ-domains of the two systems are disjoint, however, unless the connection is real analytic.

(2) Under a gauge transformation $A\to Q^{-1}AQ+Q^{-1}\mathrm{d}Q$, the function $h\to h\tau^*(Q)$. If the gauge is fixed as in the preceding section, the only freedom in h is that of left multiplication by an arbitrary holomorphic $\mathrm{GL}(n,\mathbb{C})$-valued function on P_0.

(3) The situation is quite different in the positive definite case [2], where P_0 is replaced by topologically non-trivial sets containing entire projective lines. Rather than splitting globally, the corresponding holomorphic functions $\{h_a h_b^{-1}=g_{ab}\}$ define a non-trivial holomorphic vector bundle over (all or part of) $\mathrm{P}_3(\mathbb{C})$ which, by construction, is trivial on the lines corresponding to points in $\mathbb{E}^4$. For $x\in\mathbb{E}^4$, $g_{ab}|L_x$ can be split into real analytic factors by solving the Riemann–Hilbert problem. Differentiating as in (6) above then gives a real an-

alytic connection which is gauge equivalent to the original one. This procedure is evidently not available in the case at hand.

4. The analytic case

Suppose now that the connection is both real analytic and $\mathrm{u}(n)$-valued. Then the solution $\Phi(x, \lambda)$ of section 2 is unitary and real analytic for $(x, \lambda) \in M \times S^1$. So it extends holomorphically in λ to a neighborhood U of $M \times S^1$ in $M \times S^2$, where it satisfies $\Phi^*(x, 1/\bar{\lambda}) = \Phi^{-1}(x, \lambda)$. Let $U_0 = U \cap (M \times D_0)$ be the common domain of Φ and the solution H of section 3.

The product ΦH^{-1} is annihilated by the operators $\partial_{\bar{y}} - \lambda\, \partial_z$ and $\partial_{\bar{z}} - \lambda\, \partial_y$. So, if we define g on $p(U)_0 \cap P_0$ by $p^* g = \Phi H^{-1}$, then g is holomorphic in v and w. Moreover, $\partial(\Phi H^{-1})/\partial\bar{\lambda} = 0 = (\partial_{\bar{\lambda}} + z\, \partial_{\bar{v}} + y\, \partial_{\bar{w}}) g = \partial_{\bar{\lambda}} g = 0$, so g is holomorphic in λ as well. Now $g | L_x$ is holomorphic and non-singular in an annulus of the form $\epsilon_x < |\lambda| < 1$ and evidently factors as

$$g(y + \lambda\bar{z}, z + \lambda\bar{y}, \lambda) = \Phi(x, \lambda) \cdot H^{-1}(x, \lambda) , \tag{7}$$

where both Φ and H solve the appropriate linear systems for the same SDYM field.

Conversely, suppose S is an open neighborhood of pN in $P_3(\mathbb{C})$, and set $S_0 = S \cap P_0$. Then we have

Theorem 3. *Let $g: S_0 \to \mathrm{GL}(n, \mathbb{C})$ be holomorphic.*

(a) For each x in M, $g | L_x$ factors uniquely as in (7), with Φ unitary and real analytic for $(x, \lambda) \in M \times S^1$ and $\Phi(x, 1) \equiv I$. $H(x, \lambda)$ is real analytic in x and holomorphic in λ for $|\lambda| < 1$.

(b) Both Φ and H solve the linear system for SDYM; in particular, the right-hand side of (6) is linear in λ.

The proof is given elsewhere [8], but some remarks are in order.

(1) If g extends holomorphically to pN, then $g | L_x$ is real analytic on the unit circle, and (a) is theorem 8.1.1 in Pressley and Segal [11]. In the general case, (a) follows from a combination of the cited theorem, a rescaling of λ, and the Birkhoff factorization [11]. Differentiability of the factors follows from uniqueness of the factorization.

(2) To conclude that, say, $H^{-1}(\partial_{\bar{y}} - \lambda\, \partial_z) H$ is linear in λ, one must show that it is globally defined on the Riemann sphere. This follows because

$$[\Phi(x, 1/\bar{\lambda}) \cdot H^{-1}(x, 1/\bar{\lambda})]^* =: \hat{H}(x, \lambda) \cdot \Phi^{-1}(x, \lambda)$$

is the pullback to $M \times D_\infty$ of a function $\hat{g}^{-1}$, holomorphic in the corresponding subdomain of P_∞. Since $\hat{H}$ is holomorphic in λ for $|\lambda| > 1$, differentiating both

$g|L_x$ and $\hat{g}|L_x$ gives a common (hence global) result for $H^{-1}(\partial_y - \lambda\,\partial_z)H$, $\Phi^{-1}(\partial_y - \lambda\,\partial_z)\Phi$, and $\hat{H}^{-1}(\partial_y - \lambda\,\partial_z)\hat{H}$, which must therefore be linear in λ by the usual argument [1].

5. Parametrizing the real analytic solutions

Let G be the group, under pointwise multiplication, of holomorphic maps $g: S_0 \to \mathrm{GL}(n, \mathbb{C})$, where S_0 is of the form $S \cap P_0$, for some neighborhood S of pN. Two such maps are regarded as equivalent if they agree on their common domain. For each element of G, theorem 3 gives a unique extended solution $\Phi(x, \lambda)$ to the linear system (5). The solution Φ is clearly unchanged if (and only if) g is replaced by gk, where k is non-singular and holomorphic on the whole of P_0. Denoting the subgroup of holomorphic maps from P_0 to $\mathrm{GL}(n, \mathbb{C})$ by G_+, we obtain a bijection between the set of extended solutions and points of the homogeneous space G/G_+.

As for the gauge fields themselves, we have already remarked that requiring $\Phi(x, 1) \equiv I$ uniquely fixes gauges. If two different extended solutions Φ and Ψ give the same gauge field, then necessarily $\Phi\Psi^{-1} = p^*\gamma$, where γ is real analytic and unitary on pN, and satisfies $\gamma(v, w, 1) \equiv I$. These maps also form a group, denoted $G_{\mathbb{R}}$, under pointwise multiplication. As they all extend holomorphically into P_0, we may regard $G_{\mathbb{R}}$ as a subgroup of G. We then have

Theorem 4. *The set of real analytic solutions to the SDYM equations on M with gauge group $\mathrm{U}(n)$ is parametrized by the double coset space $G_{\mathbb{R}} \backslash G / G_+$.*

Remark. We mention a peculiar consequence of the above results. Suppose the connection A is real analytic, but not skew-Hermitian, and let Ψ and K be the solutions to (5) and (6), respectively. Then ΨK^{-1} is still the pullback of a holomorphic function g on some subdomain of P_0. By theorem 3, it can be split as ΦH^{-1} (with Φ unitary on the unit circle) to get a *different* solution to the SDYM equations. In fact, $\Psi = \Phi \cdot [H^{-1}K]$ is exactly the unique factorization of a real analytic loop given by the theorem of Pressley and Segal cited above. In other words, take any real analytic solution $\Psi(x, \lambda)$ to the linear system on $M \times S^1$ and think of it as a map Ψ from M to the loop group $\mathrm{LGL}(n, \mathbb{C})$. For each x in M, let $\Phi(x, \lambda)$ be the unique projection into $\Omega\mathrm{U}(n)$ given by the factorization. Then $\Phi(x, \lambda)$ is an extended (unitary) solution to the SDYM equations in M.

References

[1] R.S. Ward, On self-dual gauge fields, Phys. Lett. A 61 (1977) 81–82.
[2] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. A 362 (1978) 425–461.
[3] L.J. Mason and G.A.J. Sparling, Nonlinear Schrödinger and KdV are reductions of self-dual Yang–Mills, Phys. Lett. A 137 (1989) 29–33.
[4] R.S. Ward, Integrable and solvable systems, and relations among them, Philos. Trans. R. Soc. A 315 (1985) 451–462.
[5] K. Uhlenbeck, Harmonic maps into Lie groups, J. Diff. Geom. 30 (1989) 1–50.
[6] R.S. Ward, Classical solutions of the chiral model, unitons, and holomorphic vector bundles, Commun. Math. Phys. 128 (1990) 319–332.
[7] R. Penrose and W. Rindler, *Spinors and Space-Time*, Vol. 2 (Cambridge Univ. Press, Cambridge, 1986).
[8] D. Lerner, Self-dual gauge fields in a spacetime of signature 0, to appear.
[9] B. Malgrange, Lectures on functions of several complex variables, Tata Institute (1958).
[10] H. Grauert and R. Remert, *Coherent Analytic Sheaves* (Springer, Berlin, 1984).
[11] A. Pressley and G. Segal, *Loop Groups* (Oxford Univ. Press, Oxford, 1986).